Optimal decay-error estimates for the hyperbolic-parabolic singular perturbation of a degenerate nonlinear equation

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Abstract

We consider a *degenerate* hyperbolic equation of Kirchhoff type with a small parameter ε in front of the second-order time-derivative.

In a recent paper, under a suitable assumption on initial data, we proved decay-error estimates for the difference between solutions of the hyperbolic problem and the corresponding solutions of the limit parabolic problem. These estimates show in the same time that the difference tends to zero both as $\varepsilon \to 0^+$, and as $t \to +\infty$. In particular, in that case the difference decays *faster* than the two terms separately.

In this paper we consider the complementary assumption on initial data, and we show that now the *optimal* decay-error estimates involve a decay rate which is *slower* than the decay rate of the two terms.

In both cases, the improvement or deterioration of decay rates depends on the smallest frequency represented in the Fourier components of initial data.

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1 Introduction

Let H be a separable real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x\rangle \geq 0$ for every $x \in D(A)$, so that for every $\alpha \geq 0$ the power $A^{\alpha}x$ is defined provided that x lies in a suitable domain $D(A^{\alpha})$.

We consider the Cauchy problem

$$\varepsilon u_{\varepsilon}''(t) + u_{\varepsilon}'(t) + |A^{1/2}u_{\varepsilon}(t)|^{2\gamma}Au_{\varepsilon}(t) = 0 \qquad \forall t \geq 0, \tag{1.1}$$

$$u_{\varepsilon}(0) = u_0, \qquad\qquad u_{\varepsilon}'(0) = u_1, \tag{1.2}$$

where $\varepsilon > 0$ and $\gamma \geq 1$ are real parameters, and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ are initial data satisfying the mild nondegeneracy condition

$$A^{1/2}u_0 \neq 0. \tag{1.3}$$

The singular perturbation problem in its generality consists in proving the convergence of solutions of (1.1), (1.2) to solutions of the first order problem

$$u'(t) + |A^{1/2}u(t)|^{2\gamma}Au(t) = 0 \qquad \forall t \geq 0, \tag{1.4}$$

$$u(0) = u_0, \tag{1.5}$$

obtained setting formally $\varepsilon = 0$ in (1.1), and omitting the second initial condition in (1.2).

This problem has generated a considerable literature in the last 30 years. In particular, it is well known that the parabolic problem (1.4), (1.5) has a unique global solution for every $u_0 \in D(A)$ (and even for less regular data), and the hyperbolic problem (1.1), (1.2) has a unique global solution provided that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the nondegeneracy assumption (1.3) and ε is small enough. The interested reader is referred to the survey [8], where more general nonlinearities and more general dissipative terms have also been considered (see also [9] for the special case of (1.1), and for more references).

Next step consists in estimating the behavior of $u(t)$, $u_{\varepsilon}(t)$, and of the difference $u_{\varepsilon}(t) - u(t)$ as $t \to +\infty$ and as $\varepsilon \to 0^+$. This gave rise to three types of results.

(A) *Decay estimates.* In this case ε is fixed, and $t \to +\infty$. The prototype of decay estimates, in the case of coercive operators, is that

$$|A^{1/2}u(t)|^2 \leq \frac{C}{(1+t)^{1/\gamma}} \quad \text{and} \quad |A^{1/2}u_{\varepsilon}(t)|^2 \leq \frac{C}{(1+t)^{1/\gamma}}$$

for every $t \geq 0$, where the constant C is independent of ε and of course also of t. As a consequence we have also that

$$|A^{1/2}(u_{\varepsilon}(t) - u(t))|^2 \leq \frac{C}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0. \tag{1.6}$$

These estimates have been obtained for the first time in [14], and then in [5] in full generality. We point out that the decay rate of $|A^{1/2}u(t)|$ and $|A^{1/2}u_\varepsilon(t)|$ coincides with the decay rate of solutions of the ordinary differential equation

$$y'(t) + |y(t)|^{2\gamma}y(t) = 0,$$

which is actually the special case of (1.4) where $H = \mathbb{R}$ and A is the identity (see also [11] for decay rates of second order ordinary differential equations).

(B) *Error estimates.* In this case t is fixed, and $\varepsilon \to 0^+$. The prototype of error estimates is that for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ one has that

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0, \tag{1.7}$$

where the constant C is once again independent of ε and t (global-in-time error estimates). It is well-known that ε^2 is the best possible convergence rate (even when looking for local-in-time error estimates), and that $D(A^{3/2}) \times D(A^{1/2})$ is the minimal requirement on initial data which guarantees this rate (even in the case of linear equations). We refer to [4] for these aspects.

(C) *Decay-error estimates.* They are the ultimate goal of the theory, since they represent the meeting point of (A) and (B). The general form of a decay-error estimate is something like

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq \omega(\varepsilon)\sigma(t),$$

where of course $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, and $\sigma(t) \to 0$ as $t \to +\infty$.

Trivial decay-error estimates can be obtained by exploiting the classical inequality $\min\{a, b\} \leq a^{1-\theta}b^\theta$, which holds true for every pair of positive real numbers a and b, and every $\theta \in [0, 1]$. Thus (1.6) and (1.7) can be interpolated by the estimates

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq C\frac{\varepsilon^{2-2\theta}}{(1+t)^{\theta/\gamma}} \qquad \forall t \geq 0, \tag{1.8}$$

for a suitable constant C independent of ε, t, θ. Nevertheless, such estimates are in general non-optimal, both with respect to the convergence rate $2-2\theta$, and with respect to the decay rate θ/γ.

Indeed, both in the case of linear equations [1], and in the case of nondegenerate Kirchhoff equations [12, 16, 17, 7], it was possible to prove decay-error estimates with the same decay rate of decay estimates, and the same convergence rate of error estimates. In the degenerate case, this leads to guess that

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq C\frac{\varepsilon^2}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0. \tag{1.9}$$

This problem was addressed for the first time in [9], where two results have been proved. The first one is a counterexample showing that (1.9) is false for general initial

data $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$. The second one is that a special assumption on initial data, which rules out the previous counterexample, yields an estimate which is even better than (1.9), namely that

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 \leq C \frac{\varepsilon^2}{(1+t)^{\delta/\gamma}} \qquad \forall t \geq 0 \tag{1.10}$$

for some $\delta > 1$. In other words, for these data the difference decays faster than the two terms separately.

In order to clarify the special assumption on initial data, let us assume for simplicity that the spectrum of A consists of a sequence of positive eigenvalues. Let us assume that v_0 is an eigenvector with respect to some eigenvalue μ. Let $\nu > \mu$, and let us assume that

$$\begin{aligned} u_0 &= v_0 + \text{components w.r.t. eigenvalues} \geq \nu, \\ u_1 &= \beta v_0 + \text{components w.r.t. eigenvalues} \geq \nu \end{aligned} \tag{1.11}$$

for some $\beta \in \mathbb{R}$ (possibly equal to 0). Then the second result in [9] is that (1.10) holds true with $\delta := \min\{2\gamma + 1, \nu/\mu\}$ (with a logarithmic correction when the two terms in the minimum are equal). Note that $\delta > 1$ in this case.

On the contrary, the counterexample presented in [9] is that (1.9) cannot be true when u_0 and u_1 are eigenvectors of A with respect to eigenvalues μ and ν, respectively, with $\nu < \mu$. In other words, in that case a deterioration of decay rates is expected. Quantifying this deterioration was stated as Open Problem 2 in section 4 of [9].

In this paper we investigate this open problem, and we find the best decay-error estimates which are true in this case. So we assume that v_0 is an eigenvector of A with respect to some eigenvalue μ, and that v_1 is an eigenvector of A with respect to some eigenvalue ν, with $\nu \leq \mu$. Then we take initial data of the form

$$\begin{aligned} u_0 &= v_0 + \text{components w.r.t. eigenvalues} > \mu, \\ u_1 &= v_1 + \text{components w.r.t. eigenvalues} > \nu. \end{aligned} \tag{1.12}$$

In this case we prove that (1.10) holds true with $\delta := \nu/\mu$, which is now less than or equal to 1. We also show that this exponent is optimal, in the sense that we cannot obtain a better decay rate if we want to keep the optimal convergence rate ε^2.

When the spectrum of A is an increasing sequence of eigenvalues (with no assumption on the dimension of eigenspaces), assumption (1.12) is complementary to assumption (1.11). This happens, for example, when A is the Dirichlet-Laplacian in a bounded open set with reasonable boundary, namely the operator involved in Kirchhoff equations in concrete form. Therefore, for these operators the problem of decay-error estimates is now fully closed by Theorem 2.5 in [9], and Theorem 2.2 of the present paper. We refer to Remark 2.5 for the details.

This paper is organized as follows. In section 2.1 we fix the notation and state our main results. In section 2.2 we present a toy model which roughly explains the deterioration of decay rates. In section 3 we prove our results.

2 Statements

2.1 Notation and statements

In the following we always assume that H is a Hilbert space, and A is a nonnegative self-adjoint (unbounded) operator on H with dense domain. We always assume that $\gamma \geq 1$ is a real number, and that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ is a pair of initial conditions satisfying the nondegeneracy condition (1.3).

Let $u(t)$ be the unique global solution of the first order problem (1.4), (1.5). Let $u_\varepsilon(t)$ be the unique global solution of the second order problem (1.1), (1.2), which exists at least for every $\varepsilon \in (0, \varepsilon_0)$ for some positive ε_0. Following the approach introduced in [13] in the linear case, we define the corrector $\theta_\varepsilon(t)$ as the solution of the second order *linear* ordinary differential equation

$$\varepsilon \theta_\varepsilon''(t) + \theta_\varepsilon'(t) = 0 \qquad \forall t \geq 0, \tag{2.1}$$

with initial data

$$\theta_\varepsilon(0) = 0, \qquad \theta_\varepsilon'(0) = u_1 + |A^{1/2}u_0|^{2\gamma} A u_0 =: \theta_0. \tag{2.2}$$

Since $\theta_0 = u_\varepsilon'(0) - u'(0)$, this corrector keeps into account the boundary layer due to the loss of one initial condition.

Finally, we define $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ in such a way that

$$u_\varepsilon(t) = u(t) + \theta_\varepsilon(t) + r_\varepsilon(t) = u(t) + \rho_\varepsilon(t) \qquad \forall t \geq 0.$$

With these notations, the singular perturbation problem consists in proving that $r_\varepsilon(t) \to 0$ or $\rho_\varepsilon(t) \to 0$ in some sense as $\varepsilon \to 0^+$. We recall that the two remainders play different roles. In particular, $r_\varepsilon(t)$ is well suited for estimating derivatives, while $\rho_\varepsilon(t)$ is used in estimates without derivatives. This distinction is essential. Indeed it is not possible to prove decay-error estimates on $A^\alpha r_\varepsilon(t)$ because it does not decay to 0 as $t \to +\infty$ (indeed $u_\varepsilon(t)$ and $u(t)$ tend to 0, while the corrector $\theta_\varepsilon(t)$ does not), and it is not possible to prove decay-error estimates on $A^\alpha \rho_\varepsilon'(t)$ because in general for $t = 0$ it does not tend to 0 as $\varepsilon \to 0^+$ (due to the loss of one initial condition).

Now let us introduce our assumptions on initial data. To this end, we need some basic facts from the spectral theory of operators, which we recall following [15].

Let E be the resolution of the identity associated with the operator A. For every measurable subset $J \subseteq [0, +\infty)$ we consider the space $H_J := \mathcal{R}(E(J))$, namely the range of the projection operator $E(J)$, which is a closed subspace of H. In the case where H admits a (finite or countable) orthonormal system $\{e_k\}$ made by eigenvalues of A, and $\{\lambda_k^2\}$ is the sequence of corresponding eigenvalues, then H_J is just the set of all $v \in H$ such that $\langle v, e_k \rangle = 0$ for every $k \in \mathbb{N}$ such that $\lambda_k^2 \notin J$.

We are now ready to introduce the class of initial data we consider in this paper.

Definition 2.1 (Assumption on initial data) Let μ and ν be two positive real numbers. We say that a pair of initial conditions $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfies the (μ, ν)-assumption if we can write $u_0 = v_0 + w_0$ and $u_1 = v_1 + w_1$, where

- v_0 and v_1 are eigenvectors of A with respect to the eigenvalues μ and ν, respectively,
- $w_0 \in H_{(\mu,+\infty)}$ and $w_1 \in H_{(\nu,+\infty)}$.

In other words, μ is the smallest frequency with respect to which u_0 has a nonzero component, and v_0 is such a component. Analogously, ν is the smallest frequency with respect to which u_1 has a nonzero component v_1.

The main result of this paper is the following.

Theorem 2.2 (Decay-error estimates) *Let H, A, γ, (u_0, u_1), ε_0, $u(t)$, $u_\varepsilon(t)$, $\theta_\varepsilon(t)$, $\rho_\varepsilon(t)$, $r_\varepsilon(t)$ be as usual.*

Let us assume that the pair (u_0, u_1) satisfies the (μ, ν)-assumption with

$$0 < \nu \leq \mu.$$

Then the following conclusions hold true with $\delta := \nu/\mu$.

(1) If in addition we assume that $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, then there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and a constant C such that for every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + \varepsilon(1+t)|r_\varepsilon'(t)|^2 \leq C\frac{\varepsilon^2}{(1+t)^{\delta/\gamma}} \qquad \forall t \geq 0,$$

$$\int_0^t (1+s)^{2\delta/\gamma}\left((1+s)|r_\varepsilon'(s)|^2 + \frac{|A^{1/2}\rho_\varepsilon(s)|^2}{1+s}\right) ds \leq C\varepsilon^2(1+t)^{\delta/\gamma} \qquad \forall t \geq 0.$$

(2) If in addition we assume that $(u_0, u_1) \in D(A^2) \times D(A)$, then there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and a constant C such that for every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$|A\rho_\varepsilon(t)|^2 + (1+t)^2|r_\varepsilon'(t)|^2 \leq C\frac{\varepsilon^2}{(1+t)^{\delta/\gamma}} \qquad \forall t \geq 0,$$

$$\int_0^t (1+s)^{2\delta/\gamma}\left((1+s)|A^{1/2}r_\varepsilon'(s)|^2 + \frac{|A\rho_\varepsilon(s)|^2}{1+s}\right) ds \leq C\varepsilon^2(1+t)^{\delta/\gamma} \qquad \forall t \geq 0.$$

As we are going to see in the proofs, the main point is obtaining the estimate on the lowest order term $\rho_\varepsilon(t)$. Quite surprisingly, all remaining estimates follow from this one through a linear argument presented in [9].

The following result shows that the estimates provided by Theorem 2.2 (at least the one on $\rho_\varepsilon(t)$, which implies all the rest) are optimal.

Theorem 2.3 (Optimality of decay-error estimates) *Let H, A, γ, (u_0, u_1), ε_0, $u(t)$, $u_\varepsilon(t)$, $\theta_\varepsilon(t)$, $\rho_\varepsilon(t)$, $r_\varepsilon(t)$ be as usual.*

Let us assume that u_0 and u_1 are eigenvectors of A with respect to the eigenvalues μ and ν, respectively, and let $\delta := \nu/\mu$. Let us assume also that

- *either $\nu < \mu$,*
- *or $\nu = \mu$, and u_0 and u_1 are orthogonal.*

Then there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$\sup_{t \geq 0} \left\{ (1+t)^{\delta/\gamma} |\rho_\varepsilon(t)|^2 \right\} \geq C\varepsilon^2. \tag{2.3}$$

We conclude with some comments about possible extensions and applications.

Remark 2.4 For the sake of simplicity, we limit ourselves to state and prove (2.3) when initial data are eigenvectors. The result is actually true for all initial data of the form (1.12). The reason is that components corresponding to higher frequencies decay faster (see [3]), and therefore the decay rate is always dictated by the smallest frequencies represented in u_0 and u_1.

Similarly, in the case where $\nu = \mu$, we could weaken the assumption that u_0 and u_1 are orthogonal by just asking that they are linearly independent. In this case the proof should be modified in an obvious way by introducing the component of u_1 orthogonal to u_0.

Remark 2.5 Let us assume that the spectrum of A consists of an increasing sequence of positive eigenvalues. Let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be any pair of initial conditions satisfying the nondegeneracy condition (1.3). Let μ and ν be the smallest eigenvalues with respect to which u_0 and u_1 have nonzero components v_0 and v_1, respectively.

- If $\nu > \mu$ (or if $u_1 = 0$, in which case ν is not well defined) we are in the situation of [9], which yields decay-error estimates with improved decay rates.
- If $\nu < \mu$ we are in the assumptions of Theorem 2.2 above, which yields decay error estimates with deteriorated decay rates, and in general nothing more because of Theorem 2.3 above.
- If $\nu = \mu$ we are for sure in the situation of Theorem 2.2, which in any case guarantees decay-error estimates without improvement or deterioration of decay rates ($\delta = 1$). If v_0 and v_1 are linearly independent, then $\delta = 1$ is optimal because of Theorem 2.3. If v_0 and v_1 are linearly dependent, then we are also in the assumptions of [9], and once again we get an improvement of decay rates.

Remark 2.6 The previous remark extends easily to operators whose spectrum is an increasing sequence of nonnegative eigenvalues *including* 0 (this is the case, for example, of the Neumann-Laplacian on a bounded interval).

Indeed in this case it is enough to separate components with respect to the kernel, where $u(t)$ is constant and $u_\varepsilon(t)$ coincides with the corrector $\theta_\varepsilon(t)$, and apply the theory of this paper in the subspace orthogonal to the kernel. We point out that, since the nonlinear term depends on $A^{1/2}u$ or $A^{1/2}u_\varepsilon$, what happens in the kernel has no influence in the orthogonal subspace.

2.2 Heuristics

In this section we present a simple calculation on ordinary differential equations that leads to guess that $|\rho_\varepsilon(t)| \sim \varepsilon(1+t)^{-\delta/(2\gamma)}$. Let us start with two simplifications.

- When ε is small enough, the parabolic equation is a good approximation of the hyperbolic one. As a consequence, after the initial layer, $u_\varepsilon(t)$ and $u(t)$ can *both* be considered as solutions of the *parabolic* equation.

- After the initial layer, for example at time $t = 1$, the difference $u_\varepsilon(t) - u(t)$ is of order ε. Actually this follows from the well-established local-in-time error estimates.

Under these simplifying assumptions, the singular perturbation problem is reduced to estimating the difference between two solutions of the *parabolic* equation whose initial data differ of order ε.

So let v_0 and v_1 be orthonormal eigenvectors of A with respect to eigenvalues μ and ν, respectively, with $0 < \nu < \mu$. Let us consider the solutions $u_0(t)$ and $u_1(t)$ of the parabolic problem with initial data $u_0(0) = v_0$, and $u_1(0) = v_0 + \varepsilon v_1$, respectively (so that at time $t = 0$ the difference is of order ε, and lies on a component corresponding to the smallest frequency).

Now it is easy to see that $u_0(t)$ is always a multiple of v_0, while $u_1(t)$ can be written in the form $u_1(t) := w(t)v_0 + v(t)v_1$, where $v(t)$ and $w(t)$ are solutions of the system of ordinary differential equations

$$\begin{cases} w'(t) + \mu\left(\nu v^2(t) + \mu w^2(t)\right)^\gamma w(t) = 0, \\ v'(t) + \nu\left(\nu v^2(t) + \mu w^2(t)\right)^\gamma v(t) = 0, \end{cases}$$

with initial data $w(0) = 1$, and $v(0) = \varepsilon$.

Now we make a further simplifying assumption, namely that $u_1(t) - u_0(t) \sim v(t)v_1$, which is reasonable if we accept that components corresponding to lower frequencies decay more slowly. Thus we have reduced ourselves to estimating $v(t)$.

According to the main trick introduced in [10], $v(t)$ and $w(t)$ can be written in the form

$$v(t) := \varepsilon\psi_\varepsilon(t), \qquad w(t) := [\psi_\varepsilon(t)]^{1/\delta},$$

where as usual $\delta := \nu/\mu$, and $\psi_\varepsilon(t)$ solves

$$\psi_\varepsilon'(t) + \nu\left(\nu\varepsilon^2\psi_\varepsilon^2(t) + \mu\psi_\varepsilon^{2/\delta}(t)\right)^\gamma \psi_\varepsilon(t) = 0, \qquad \psi_\varepsilon(0) = 1. \tag{2.4}$$

Now we make the final simplifying assumption by setting $\varepsilon = 0$ in (2.4), so that from now on $\psi_\varepsilon(t)$ does not depend on ε and solves

$$\psi'(t) + k\psi^{1+2\gamma/\delta}(t) = 0, \qquad \psi(0) = 1,$$

for a suitable positive constant k. This differential equation can be easily integrated, giving that $\psi(t) \sim (1+t)^{-\delta/(2\gamma)}$, hence

$$|\rho_\varepsilon(t)| \sim |u_1(t) - u_0(t)| \sim |v(t)| = \varepsilon\psi(t) \sim \varepsilon(1+t)^{-\delta/(2\gamma)},$$

which is consistent both with Theorem 2.2, and with Theorem 2.3.

We conclude with some remarks. First of all, (2.4) is more or less the same equation which appears in the proof of Theorem 2.3 (see also Lemma 3.2), which means that the rough calculation we did is actually close to reality.

Secondly, neglecting the term with ε in (2.4) is not reasonable for all times. Indeed $\psi_\varepsilon(t) \to 0$ as $t \to +\infty$, and in this regime the term with ψ_ε^2 becomes dominant over $\psi_\varepsilon^{2/\delta}$ (because $\delta < 1$). This means that as $t \to +\infty$ the true approximation of (2.4) is

$$\psi_\varepsilon'(t) + k\varepsilon^{2\gamma}\psi_\varepsilon^{1+2\gamma}(t) = 0 \qquad \psi_\varepsilon(0) = 1,$$

which gives $\psi(t) \sim (1+\varepsilon^{2\gamma}t)^{-1/(2\gamma)}$, namely a better decay rate.

Which is the correct approximation of (2.4)? In a certain sense both! The fact that $|\rho_\varepsilon(t)| \sim \varepsilon(1+\varepsilon^{2\gamma}t)^{-1/(2\gamma)}$, suggested by the second approximation, is consistent with previous works where it was proved that both $u(t)$ and $u_\varepsilon(t)$ decay as $(1+t)^{-1/(2\gamma)}$. The point is that the final coefficient in this case turns out to be of order 1, not of order ε, and therefore we get a better decay rate which we pay by losing the convergence rate. The first approximation, on the contrary, preserves the optimal convergence rate of order ε. As in (1.8), one could interpolate the two extremes by finding a family of estimates with intermediate decay rates and intermediate convergence rates.

The interplay between the two different regimes of (2.4) is what makes highly nontrivial the rigorous analysis of decay-error estimates for nonlinear degenerate equations.

3 Proofs

In the sequel we set

$$c(t) := |A^{1/2}u(t)|^{2\gamma}, \qquad c_\varepsilon(t) := |A^{1/2}u_\varepsilon(t)|^{2\gamma}, \tag{3.1}$$

and we consider their anti-derivatives

$$C(t) := \int_0^t c(s)\,ds, \qquad C_\varepsilon(t) := \int_0^t c_\varepsilon(s)\,ds. \tag{3.2}$$

We exploit that the corrector $\theta_\varepsilon(t)$, which is the solution of (2.1), (2.2), is given by the explicit formula

$$\theta_\varepsilon(t) = \varepsilon\theta_0\left(1 - e^{-t/\varepsilon}\right) \qquad \forall t \geq 0. \tag{3.3}$$

We also set

$$g_\varepsilon(t) := -(c_\varepsilon(t) - c(t))Au(t) - \varepsilon u''(t), \tag{3.4}$$

so that we can regard $\rho_\varepsilon(t)$ as the solution of the linear equation

$$\varepsilon\rho_\varepsilon''(t) + \rho_\varepsilon'(t) + c_\varepsilon(t)A\rho_\varepsilon(t) = g_\varepsilon(t), \tag{3.5}$$

with initial data

$$\rho_\varepsilon(0) = 0, \qquad \rho_\varepsilon'(0) = \theta_0, \tag{3.6}$$

and $r_\varepsilon(t)$ as the solution of the linear equation

$$\varepsilon r_\varepsilon''(t) + r_\varepsilon'(t) + c_\varepsilon(t)A\rho_\varepsilon(t) = g_\varepsilon(t), \tag{3.7}$$

with initial data

$$r_\varepsilon(0) = 0, \qquad r_\varepsilon'(0) = 0. \tag{3.8}$$

In many points we need to consider components of $u_\varepsilon(t)$ or $u(t)$ with respect to v_0 and v_1, where v_0 and v_1 are the eigenvectors of A which appear in the (μ, ν)-assumption on initial data. To this end we set

$$u_{\varepsilon,\nu}(t) := \frac{1}{|v_1|}\langle u_\varepsilon(t), v_1\rangle, \qquad\qquad u_{\varepsilon,\mu}(t) := \frac{1}{|v_0|}\langle u_\varepsilon(t), v_0\rangle. \tag{3.9}$$

It is easy to see that these real functions satisfy, respectively, the following second order equations

$$\varepsilon u_{\varepsilon,\nu}''(t) + u_{\varepsilon,\nu}'(t) + \nu c_\varepsilon(t)u_{\varepsilon,\nu}(t) = 0, \qquad \varepsilon u_{\varepsilon,\mu}''(t) + u_{\varepsilon,\mu}'(t) + \mu c_\varepsilon(t)u_{\varepsilon,\mu}(t) = 0. \tag{3.10}$$

3.1 Previous results

Many estimates on $u(t)$ and $u_\varepsilon(t)$ have been proved in literature. We refer to [5] (see also Theorems A and B in [9]) for a complete list. For the convenience of the reader, in next result we limit ourselves to recall just those estimates needed in the sequel.

Theorem A *Let H, A, γ, (u_0, u_1), ε_0, $u(t)$, $u_\varepsilon(t)$, $\theta_\varepsilon(t)$, $\rho_\varepsilon(t)$, $r_\varepsilon(t)$ be as usual. Let us assume that u_0 and u_1 are in $H_{[\nu,+\infty)}$ for some $\nu > 0$ (which is equivalent to asking the coercivity of A).*

Then there exists $\varepsilon_1 > 0$, and positive constants $M_1, \ldots, M_{16}$, such that the following estimates hold true for every $\varepsilon \in (0, \varepsilon_1)$.

(1) (Decay estimates for the parabolic problem) *We have that*

$$|A^{1/2}u(t)|^2 \leq \frac{M_1}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0, \tag{3.11}$$

$$|Au(t)|^2 \leq \frac{M_2}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0, \tag{3.12}$$

and as a consequence

$$c(t) \leq \frac{M_3}{1+t} \qquad \forall t \geq 0. \tag{3.13}$$

If in addition $u_0 \in D(A^{3/2})$, then for every $\delta \in (0, 2\gamma+1]$ we have that

$$|A^{3/2}u(t)|^2 \leq \frac{M_4}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0, \tag{3.14}$$

$$\int_0^t (1+s)^{1+2\delta/\gamma}|u''(s)|^2\, ds \leq M_5(1+t)^{\delta/\gamma} \qquad \forall t \geq 0. \tag{3.15}$$

If in addition $u_0 \in D(A^2)$, then for every $\delta \in (0, 2\gamma+1]$ we have that

$$\int_0^t (1+s)^{1+2\delta/\gamma}|A^{1/2}u''(s)|^2\, ds \leq M_6(1+t)^{\delta/\gamma} \qquad \forall t \geq 0, \tag{3.16}$$

$$|u''(t)|^2 \leq \frac{M_7}{(1+t)^{4+1/\gamma}} \qquad \forall t \geq 0. \tag{3.17}$$

(2) (Decay estimates for the hyperbolic problem) *We have that*

$$\frac{M_8}{(1+t)^{1/\gamma}} \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{M_9}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0, \tag{3.18}$$

$$|Au_\varepsilon(t)|^2 \leq \frac{M_{10}}{(1+t)^{1/\gamma}} \qquad \forall t \geq 0, \tag{3.19}$$

$$|u'_\varepsilon(t)|^2 \leq \frac{M_{11}}{(1+t)^{2+1/\gamma}} \qquad \forall t \geq 0, \tag{3.20}$$

and as a consequence

$$\frac{M_{12}}{1+t} \leq c_\varepsilon(t) \leq \frac{M_{13}}{1+t} \qquad \forall t \geq 0, \tag{3.21}$$

$$\frac{|c'_\varepsilon(t)|}{c_\varepsilon(t)} \leq \frac{M_{14}}{1+t} \qquad \forall t \geq 0. \tag{3.22}$$

(3) (Basic error estimates) *If in addition* $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, *we have that*

$$|\rho_\varepsilon(t)|^2 \leq M_{15}\varepsilon^2 \qquad \forall t \geq 0, \tag{3.23}$$

$$\int_0^{+\infty} (1+t)|r_\varepsilon'(t)|^2\,dt \leq M_{16}\varepsilon^2. \tag{3.24}$$

(4) ("Monotonicity" estimates) *We have that*

$$\langle c_\varepsilon(t)Au_\varepsilon(t) - c(t)Au(t), \rho_\varepsilon(t)\rangle \geq \frac{1}{2}\left(c_\varepsilon(t) + c(t)\right)|A^{1/2}\rho_\varepsilon(t)|^2 \qquad \forall t \geq 0. \tag{3.25}$$

We refer to [5] and to Lemma 3.3 in [9] for the estimates on solutions of the parabolic and hyperbolic problem, to Theorem 2.4 in [2] (see also [6]) for the basic error estimates, and to Lemma 3.4 in [2] for the monotonicity estimate (which actually is a general property of vectors). We point out that eigenvalues and components play no role in Theorem A above.

The situation is different in the following result, where $u_\varepsilon(t)$ and its components $u_{\varepsilon,\nu}(t)$ and $u_{\varepsilon,\mu}(t)$ are estimated in terms of $C_\varepsilon(t)$. For a proof, we refer to Theorem 3.1 in [3], where similar estimates have been obtained more generally for the components of $u_\varepsilon(t)$ in the subspaces $H_{[\nu,+\infty)}$ and $H_{[\mu,+\infty)}$.

Theorem B *Let* H, A, γ, (u_0, u_1), ε_0, $u_\varepsilon(t)$ *be as usual. Let us assume that the pair* (u_0, u_1) *satisfies the* (μ, ν)*-assumption with respect to some* $0 < \nu \leq \mu$, *let* $u_{\varepsilon,\nu}(t)$ *and* $u_{\varepsilon,\mu}(t)$ *be the components of* $u_\varepsilon(t)$, *defined according to (3.9), and let* $C_\varepsilon(t)$ *be defined according to (3.2).*

Then there exists $\varepsilon_1 > 0$, *and constants* M_{17}, M_{18}, M_{19} *such that for every* $\varepsilon \in (0, \varepsilon_1)$ *we have that*

$$|A^{1/2}u_\varepsilon(t)|^2 \leq M_{17}e^{-2\nu C_\varepsilon(t)} \qquad \forall t \geq 0, \tag{3.26}$$

$$|u_{\varepsilon,\nu}(t)|^2 + \frac{|u_{\varepsilon,\nu}'(t)|^2}{[c_\varepsilon(t)]^2} \leq M_{18}e^{-2\nu C_\varepsilon(t)} \qquad \forall t \geq 0, \tag{3.27}$$

$$|u_{\varepsilon,\mu}(t)|^2 + \frac{|u_{\varepsilon,\mu}'(t)|^2}{[c_\varepsilon(t)]^2} \leq M_{19}e^{-2\mu C_\varepsilon(t)} \qquad \forall t \geq 0. \tag{3.28}$$

The last result we need allows to deduce all the conclusions of Theorem 2.2 from the only estimate on $\rho_\varepsilon(t)$. It is actually a result on linear equations, in the sense that now we regard $\rho_\varepsilon(t)$ and $r_\varepsilon(t)$ as solutions of (3.5) and (3.7), forgetting that $c(t)$, $c_\varepsilon(t)$, and $g_\varepsilon(t)$ are given by (3.1) and (3.4), respectively. For a proof, we refer to Proposition 3.5 in [9].

Proposition C *Let* H *and* A *be as usual, and let* ε_0, γ, δ *be positive real numbers.*

For every $\varepsilon \in (0, \varepsilon_0)$, *let us assume that* $\rho_\varepsilon(t)$, $r_\varepsilon(t)$, $c_\varepsilon(t)$, *and* $g_\varepsilon(t)$ *satisfy (3.5) through (3.8). Moreover, let us assume that*

(i) the solution $\rho_\varepsilon(t)$ satisfies the a priori estimate

$$|\rho_\varepsilon(t)|^2 \leq M_{20}\frac{\varepsilon^2}{(1+t)^{\delta/\gamma}} \qquad \forall t \geq 0, \tag{3.29}$$

(ii) the coefficient $c_\varepsilon : [0, +\infty) \to (0, +\infty)$ is of class C^1 and satisfies (3.21) and (3.22),

(iii) the forcing term $g_\varepsilon : [0, +\infty) \to H$ is continuous and such that

$$\int_0^t (1+s)^{1+2\delta/\gamma}|g_\varepsilon(s)|^2\, ds \leq M_{21}\varepsilon^2(1+t)^{\delta/\gamma} \qquad \forall t \geq 0. \tag{3.30}$$

Then the following conclusions hold true.

(1) If $\theta_0 \in D(A^{1/2})$, then all the estimates in statement (1) of Theorem 2.2 hold true.

(2) If in addition we have that $\theta_0 \in D(A)$, and $g_\varepsilon(t)$ satisfies also

$$\int_0^t (1+s)^{1+2\delta/\gamma}|A^{1/2}g_\varepsilon(s)|^2\, ds \leq M_{22}\varepsilon^2(1+t)^{\delta/\gamma} \qquad \forall t \geq 0, \tag{3.31}$$

$$|g_\varepsilon(t)|^2 \leq M_{23}\frac{\varepsilon^2}{(1+t)^{2+\delta/\gamma}} \qquad \forall t \geq 0, \tag{3.32}$$

then all the estimates in statement (2) of Theorem 2.2 hold true.

3.2 Preliminary estimates

In the following result, we collect some further estimates on solutions of the hyperbolic problem. Most of them have been already used somewhere in previous papers, but for the convenience of the reader we give here a self contained proof (of course based on the estimates of the previous section).

Proposition 3.1 *Let us consider the same assumptions of Theorem 2.2, and let $C(t)$ and $C_\varepsilon(t)$ be defined by (3.2).*

Then there exists $\varepsilon_1 \in (0, \varepsilon_0)$, and positive constants M_{24}, ..., M_{27}, such that for every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$e^{2\mu\gamma C(t)} \leq M_{24}(1+t) \qquad \forall t \geq 0, \tag{3.33}$$

$$e^{2\nu\gamma C_\varepsilon(t)} \leq M_{25}(1+t) \qquad \forall t \geq 0, \tag{3.34}$$

$$e^{2\mu\gamma C(t)} \geq M_{26}(1+t) \qquad \forall t \geq 0, \tag{3.35}$$

$$e^{2\mu\gamma C_\varepsilon(t)} \geq M_{27}(1+t) \qquad \forall t \geq 0. \tag{3.36}$$

Proof Let us prove the four estimates separately.

Proof of estimate (3.33) From (1.4) we have that

$$\left[|A^{1/2}u(t)|^2 e^{2\mu C(t)}\right]' = 2\mu c(t) e^{2\mu C(t)} |A^{1/2}u(t)|^2 - 2e^{2\mu C(t)} c(t) |Au(t)|^2. \tag{3.37}$$

On the other hand, $u(t) \in H_{[\mu,+\infty)}$ for every $t \geq 0$, hence $|Au(t)|^2 \geq \mu |A^{1/2}u(t)|^2$. It follows that the right-hand side of (3.37) is less than or equal to 0, hence

$$|A^{1/2}u(t)|^2 e^{2\mu C(t)} \leq |A^{1/2}u_0|^2.$$

Therefore we have that

$$\left[e^{2\mu\gamma C(t)}\right]' = 2\mu\gamma c(t) e^{2\mu\gamma C(t)} = 2\mu\gamma \left(|A^{1/2}u(t)|^2 e^{2\mu C(t)}\right)^{\gamma} \leq k_1,$$

so that (3.33) follows by integration.

Proof of estimate (3.34) Thanks to (3.26) we have that

$$\left[e^{2\nu\gamma C_\varepsilon(t)}\right]' = 2\nu\gamma c_\varepsilon(t) e^{2\nu\gamma C_\varepsilon(t)} = 2\nu\gamma \left(|A^{1/2}u_\varepsilon(t)|^2 e^{2\nu C_\varepsilon(t)}\right)^{\gamma} \leq k_2,$$

so that (3.34) follows by integration.

Proof of estimate (3.35) Let us begin by showing that

$$|A^{1/2}u(t)|^2 e^{2\mu C(t)} \geq k_3 > 0 \qquad \forall t \geq 0. \tag{3.38}$$

To this end, let v_0 denote the component of u_0 with respect to the eigenspace of μ (as in Definition 2.1), and let $u_\mu(t) := |v_0|^{-1}\langle u(t), v_0\rangle$ be the component of $u(t)$ with respect to the same eigenspace. Then of course $u_\mu(t)$ satisfies

$$u_\mu'(t) + \mu c(t) u_\mu(t) = 0, \qquad u_\mu(0) = |v_0|,$$

hence $u_\mu(t) = |v_0| e^{-\mu C(t)}$ for every $t \geq 0$. Therefore we have that

$$|A^{1/2}u(t)|^2 e^{2\mu C(t)} \geq \mu |u_\mu(t)|^2 e^{2\mu C(t)} \geq \mu |v_0|^2,$$

which implies (3.38). It follows that

$$\left[e^{2\mu\gamma C(t)}\right]' = 2\mu\gamma c(t) e^{2\mu\gamma C(t)} = 2\mu\gamma \left(|A^{1/2}u(t)|^2 e^{2\mu C(t)}\right)^{\gamma} \geq k_4 > 0,$$

so that (3.35) follows by integration.

Proof of estimate (3.36) As before, we begin by showing that

$$|A^{1/2}u_\varepsilon(t)|^2 e^{2\mu C_\varepsilon(t)} \geq k_5 > 0 \qquad \forall t \geq 0. \tag{3.39}$$

To this end, let v_0 be the projection of u_0 in the eigenspace of μ, let $u_{\varepsilon,\mu}(t)$ be defined as in (3.9), and let $y_\varepsilon(t) := [u_{\varepsilon,\mu}(t)]^2$. From (3.10) we have that

$$y_\varepsilon'(t) = -2\mu c_\varepsilon(t) y_\varepsilon(t) - 2\varepsilon\, u_{\varepsilon,\mu}(t) \cdot u_{\varepsilon,\mu}''(t).$$

Since $y_\varepsilon(0) = |v_0|^2$, integrating this differential equation we obtain that

$$e^{2\mu C_\varepsilon(t)} y_\varepsilon(t) = |v_0|^2 - 2\varepsilon \int_0^t u_{\varepsilon,\mu}(s) \cdot u''_{\varepsilon,\mu}(s) \cdot e^{2\mu C_\varepsilon(s)}\, ds. \tag{3.40}$$

In order to estimate the last term, we integrate by parts, and we find that

$$\begin{aligned}
\int_0^t u_{\varepsilon,\mu}(s) \cdot u''_{\varepsilon,\mu}(s) \cdot e^{2\mu C_\varepsilon(s)}\, ds &= u_{\varepsilon,\mu}(t) \cdot u'_{\varepsilon,\mu}(t) \cdot e^{2\mu C_\varepsilon(t)} - u_{\varepsilon,\mu}(0) \cdot u'_{\varepsilon,\mu}(0) \\
&\quad - \int_0^t [u'_{\varepsilon,\mu}(s)]^2 e^{2\mu C_\varepsilon(s)}\, ds - \int_0^t u_{\varepsilon,\mu}(s) \cdot u'_{\varepsilon,\mu}(s) \cdot 2\mu c_\varepsilon(s) e^{2\mu C_\varepsilon(s)}\, ds \\
&=: A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

Let us estimate the four terms separately. From (3.28) we have that

$$|u_{\varepsilon,\mu}(t)| \leq k_6 e^{-\mu C_\varepsilon(t)} \qquad \forall t \geq 0, \tag{3.41}$$

$$|u'_{\varepsilon,\mu}(t)| \leq k_7 c_\varepsilon(t) e^{-\mu C_\varepsilon(t)} \qquad \forall t \geq 0. \tag{3.42}$$

Therefore, the estimate from above in (3.21) implies that

$$A_1 \leq |u_{\varepsilon,\mu}(t)| \cdot |u'_{\varepsilon,\mu}(t)| \cdot e^{2\mu C_\varepsilon(t)} \leq k_8.$$

The term A_2 is a constant, independent of ε, and of course $A_3 \leq 0$.
As for A_4, exploiting once more (3.41) and (3.42), we have that

$$-u_{\varepsilon,\mu}(t) \cdot u'_{\varepsilon,\mu}(t) \leq |u_{\varepsilon,\mu}(t)| \cdot |u'_{\varepsilon,\mu}(t)| \leq k_9 c_\varepsilon(t) e^{-2\mu C_\varepsilon(t)}.$$

Therefore, the estimate from above in (3.21) implies that

$$A_4 \leq k_{10} \int_0^t [c_\varepsilon(s)]^2 ds \leq k_{11} \int_0^t \frac{1}{(1+s)^2}\, ds \leq k_{11}.$$

In conclusion, we have proved that $A_1 + A_2 + A_3 + A_4 \leq k_{12}$. Coming back to (3.40), we have obtained that

$$e^{2\mu C_\varepsilon(t)} y_\varepsilon(t) \geq |v_0|^2 - 2k_{12}\varepsilon \geq \frac{|v_0|^2}{2} \qquad \forall t \geq 0,$$

provided that ε is small enough. Therefore we have that

$$|A^{1/2} u_\varepsilon(t)|^2 e^{2\mu C_\varepsilon(t)} \geq \mu |u_{\varepsilon,\mu}(t)|^2 e^{2\mu C_\varepsilon(t)} \geq \mu \frac{|v_0|^2}{2},$$

which implies (3.39). It follows that

$$\left[e^{2\mu\gamma C_\varepsilon(t)}\right]' = 2\mu\gamma c_\varepsilon(t) e^{2\mu\gamma C_\varepsilon(t)} = 2\mu\gamma \left(|A^{1/2} u_\varepsilon(t)|^2 e^{2\mu C_\varepsilon(t)}\right)^\gamma \geq k_{13} > 0$$

so that (3.36) follows by integration. $\square$

Next result is an estimate for a supersolution of an ordinary differential equation.

Lemma 3.2 *Let $0 < \delta \leq 1$ and $K > 0$ be two real numbers.*

Then there exist $\varepsilon_1 > 0$ and $M_{28} > 0$, both depending on δ and K, such that the following property holds true. For every $\varepsilon \in (0, \varepsilon_1)$, and every function $\psi_\varepsilon \in C^1([0, +\infty), \mathbb{R})$ such that $\psi_\varepsilon(0) = 1$, and

$$\psi_\varepsilon'(t) \geq -K\psi_\varepsilon(t) \left(\varepsilon^{2\gamma}[\psi_\varepsilon(t)]^{2\gamma} + [\psi_\varepsilon(t)]^{2\gamma/\delta}\right) \qquad \forall t \geq 0, \tag{3.43}$$

we have that

$$\psi_\varepsilon\left(\frac{1}{\varepsilon^\delta}\right) \geq M_{28}\varepsilon^{\delta^2/(2\gamma)}. \tag{3.44}$$

Proof Let us consider the differential equation

$$y' = -Ky\left\{\varepsilon^{2\gamma}y^{2\gamma} + y^{2\gamma/\delta}\right\}. \tag{3.45}$$

Assumption (3.43) is equivalent to saying that $\psi_\varepsilon(t)$ is a supersolution of (3.45) for every $t \geq 0$. Let us set

$$z(t) := \left(\frac{\delta}{4K\gamma t + \delta}\right)^{\delta/(2\gamma)} \qquad \forall t \geq 0.$$

We claim that, for ε small enough, $z(t)$ is a subsolution of (3.45) for $t \in [0, 1/\varepsilon^\delta]$. Indeed this is equivalent to showing that

$$z'(t) = -2K[z(t)]^{1+2\gamma/\delta} \leq -Kz(t)\left\{\varepsilon^{2\gamma}[z(t)]^{2\gamma} + [z(t)]^{2\gamma/\delta}\right\} \qquad \forall t \in [0, 1/\varepsilon^\delta],$$

which in turn is equivalent to

$$[z(t)]^{(1-\delta)/\delta} \geq \varepsilon \qquad \forall t \in [0, 1/\varepsilon^\delta]. \tag{3.46}$$

Since $z(t)$ is decreasing, and $0 < \delta \leq 1$, it is enough to check (3.46) when $t = 1/\varepsilon^\delta$. Now for ε small enough we have that

$$\left[z\left(\frac{1}{\varepsilon^\delta}\right)\right]^{(1-\delta)/\delta} = \left[\frac{\delta\varepsilon^\delta}{4K\gamma + \delta\varepsilon^\delta}\right]^{(1-\delta)/(2\gamma)} \geq k_1\varepsilon^{\delta(1-\delta)/(2\gamma)}. \tag{3.47}$$

Since $\delta(1-\delta)/(2\gamma) < 1$, inequality (3.47) implies (3.46) when ε is small enough.

This proves that $z(t)$ is a subsolution of (3.45) in the given interval. Since $z(0) = 1 = \psi_\varepsilon(0)$, the usual comparison principle yields that

$$\psi_\varepsilon\left(\frac{1}{\varepsilon^\delta}\right) \geq z\left(\frac{1}{\varepsilon^\delta}\right) \geq k_2\varepsilon^{\delta^2/(2\gamma)},$$

which proves (3.44). $\square$

3.3 Proof of Theorem 2.2

This proof is organized as the proof of the main result of [9]. The first part is the nonlinear core of the argument, where we prove that $\rho_\varepsilon(t)$ satisfies (3.29), of course under the assumption that $(u_0,u_1)\in D(A^{3/2})\times D(A^{1/2})$. In the second part we apply Proposition C in order to deduce all the conclusions of Theorem 2.2.

3.3.1 Nonlinear core

Let us set $y_\varepsilon(t):=|\rho_\varepsilon(t)|^2$. From (3.5) and (3.4) we have that

$$y_\varepsilon'(t)=-2\langle c_\varepsilon(t)Au_\varepsilon(t)-c(t)Au(t),\rho_\varepsilon(t)\rangle-2\varepsilon\langle u_\varepsilon''(t),\rho_\varepsilon(t)\rangle,$$

hence by (3.25) it follows that

$$y_\varepsilon'(t)\le-(c_\varepsilon(t)+c(t))|A^{1/2}\rho_\varepsilon(t)|^2-2\varepsilon\langle u_\varepsilon''(t),\rho_\varepsilon(t)\rangle. \tag{3.48}$$

Since $u(t)$ and $u_\varepsilon(t)$, hence also $\rho_\varepsilon(t)$, lie in the subspace $H_{[\nu,+\infty)}$, we have that $|A^{1/2}\rho_\varepsilon(t)|^2\ge\nu|\rho_\varepsilon(t)|^2$. Thus (3.48) implies that

$$y_\varepsilon'(t)\le-\nu(c_\varepsilon(t)+c(t))y_\varepsilon(t)-2\varepsilon\langle u_\varepsilon''(t),\rho_\varepsilon(t)\rangle.$$

Since $y_\varepsilon(0)=0$, integrating this differential inequality we obtain that

$$y_\varepsilon(t)\le-2\varepsilon e^{-\nu(C_\varepsilon(t)+C(t))}\int_0^t\langle u_\varepsilon''(s),\rho_\varepsilon(s)\rangle e^{\nu(C_\varepsilon(s)+C(s))}\,ds. \tag{3.49}$$

In order to estimate the last term, we integrate by parts, and we find that

$$\begin{aligned}-2\varepsilon\int_0^t\langle u_\varepsilon''(s),\rho_\varepsilon(s)\rangle e^{\nu(C_\varepsilon(s)+C(s))}\,ds &= -2\varepsilon\langle u_\varepsilon'(t),\rho_\varepsilon(t)\rangle e^{\nu(C_\varepsilon(t)+C(t))}\\ &\quad+2\varepsilon\int_0^t\langle u_\varepsilon'(s),\rho_\varepsilon'(s)\rangle e^{\nu(C_\varepsilon(s)+C(s))}\,ds\\ &\quad+2\varepsilon\int_0^t\langle u_\varepsilon'(s),\rho_\varepsilon(s)\rangle\nu(c_\varepsilon(s)+c(s))e^{\nu(C_\varepsilon(s)+C(s))}\,ds\\ &=: A_1+A_2+A_3.\end{aligned}$$

Let us estimate the three terms separately. From (3.20) we have that

$$2\varepsilon|u_\varepsilon'(t)|\cdot|\rho_\varepsilon(t)|\le2\varepsilon^2|u_\varepsilon'(t)|^2+\frac{1}{2}|\rho_\varepsilon(t)|^2\le\frac{2k_1\varepsilon^2}{(1+t)^{2+1/\gamma}}+\frac{1}{2}|\rho_\varepsilon(t)|^2,$$

hence

$$A_1\le2\varepsilon|u_\varepsilon'(t)|\cdot|\rho_\varepsilon(t)|\cdot e^{\nu(C_\varepsilon(t)+C(t))}\le\left(\frac{2k_1\varepsilon^2}{(1+t)^{2+1/\gamma}}+\frac{1}{2}|\rho_\varepsilon(t)|^2\right)e^{\nu(C_\varepsilon(t)+C(t))}. \tag{3.50}$$

In order to estimate A_2, we first observe that

$$2\varepsilon\langle u_\varepsilon'(t), \rho_\varepsilon'(t)\rangle \leq 2\varepsilon|u_\varepsilon'(t)| \cdot (|r_\varepsilon'(t)| + |\theta_\varepsilon'(t)|) \leq \varepsilon^2|u_\varepsilon'(t)|^2 + |r_\varepsilon'(t)|^2 + 2\varepsilon|u_\varepsilon'(t)| \cdot |\theta_\varepsilon'(t)|.$$

Therefore, from (3.20) and the explicit formula (3.3) for $\theta_\varepsilon(t)$, we obtain that

$$2\varepsilon\langle u_\varepsilon'(t), \rho_\varepsilon'(t)\rangle \leq k_1\frac{\varepsilon^2}{(1+t)^{2+1/\gamma}} + |r_\varepsilon'(t)|^2 + k_2\frac{\varepsilon}{(1+t)^{1+1/(2\gamma)}} \cdot e^{-t/\varepsilon}.$$

On the other hand, from (3.33), (3.34), and the fact that $\nu \leq \mu$, we have that

$$e^{\nu(C_\varepsilon(t)+C(t))} \leq e^{\nu C_\varepsilon(t)} \cdot e^{\mu C(t)} \leq k_3(1+t)^{1/\gamma}, \tag{3.51}$$

and therefore

$$\begin{aligned}
2\varepsilon\langle u_\varepsilon'(t), \rho_\varepsilon'(t)\rangle e^{\nu(C_\varepsilon(t)+C(t))} &\leq k_4\frac{\varepsilon^2}{(1+t)^2} + k_3|r_\varepsilon'(t)|^2(1+t)^{1/\gamma} + k_5\frac{\varepsilon}{(1+t)^{1-1/(2\gamma)}}e^{-t/\varepsilon} \\
&\leq k_4\frac{\varepsilon^2}{(1+t)^2} + k_3|r_\varepsilon'(t)|^2(1+t) + k_5\varepsilon e^{-t/\varepsilon}.
\end{aligned}$$

Integrating in $[0,t]$, and exploiting (3.24), we obtain that

$$A_2 \leq k_6\varepsilon^2. \tag{3.52}$$

In order to estimate A_3, we first apply (3.13), (3.21), and (3.51) in order to obtain that

$$2\varepsilon\langle u_\varepsilon'(t), \rho_\varepsilon(t)\rangle \nu(c_\varepsilon(t)+c(t))e^{\nu(C_\varepsilon(t)+C(t))} \leq k_7\varepsilon|u_\varepsilon'(t)| \cdot |\rho_\varepsilon(t)| \cdot \frac{1}{1+t} \cdot (1+t)^{1/\gamma}.$$

Now we estimate $|u_\varepsilon'(t)|$ and $|\rho_\varepsilon(t)|$ by means of (3.20) and (3.23), respectively. We obtain that

$$2\varepsilon\langle u_\varepsilon'(t), \rho_\varepsilon(t)\rangle \nu(c_\varepsilon(t)+c(t))e^{\nu(C_\varepsilon(t)+C(t))} \leq k_8\frac{\varepsilon^2}{(1+t)^{2-1/(2\gamma)}}.$$

Since $2 - 1/(2\gamma) > 1$, integrating in $[0,t]$ we obtain that

$$A_3 \leq k_9\varepsilon^2. \tag{3.53}$$

Plugging (3.50), (3.52), and (3.53) into (3.49), we have that

$$y_\varepsilon(t) \leq k_{10}\frac{\varepsilon^2}{(1+t)^{2+1/\gamma}} + \frac{1}{2}y_\varepsilon(t) + k_{11}\varepsilon^2 e^{-\nu(C_\varepsilon(t)+C(t))} \tag{3.54}$$

Finally, (3.35) and (3.36) imply that

$$e^{-\nu(C_\varepsilon(t)+C(t))} = \left(e^{-\mu C_\varepsilon(t)} \cdot e^{-\mu C(t)}\right)^{\nu/\mu} \leq k_{12}\frac{1}{(1+t)^{\delta/\gamma}}. \tag{3.55}$$

Since $2 + 1/\gamma \geq 1/\gamma \geq \delta/\gamma$, plugging (3.55) into (3.54) we find that

$$\frac{1}{2}y_\varepsilon(t) \leq k_{13}\frac{\varepsilon^2}{(1+t)^{\delta/\gamma}},$$

which is equivalent to (3.29).

3.3.2 Linear conclusion

It remains to show that the assumptions of Proposition C are satisfied. The argument is the same as in section 3.5 of [9], with the obvious changes in the exponents.

The a priori estimate on $|\rho_\varepsilon(t)|$ is exactly the content of the nonlinear core. The assumptions on $c_\varepsilon(t)$ are exactly (3.21) and (3.22).

In order to prove estimates on $g_\varepsilon(t)$, we first estimate $c_\varepsilon(t) - c(t)$. To this end, we apply the mean value theorem to the function σ^γ, and we obtain the inequality

$$|y^\gamma - x^\gamma| \leq \gamma \max\{y^{\gamma-1}, x^{\gamma-1}\} \cdot |y - x| \qquad \forall x \geq 0,\ \forall y \geq 0.$$

Setting $y := |A^{1/2}u_\varepsilon(t)|^2$ and $x := |A^{1/2}u(t)|^2$, it follows that

$$|c_\varepsilon(t) - c(t)| \leq \gamma \max\left\{|A^{1/2}u_\varepsilon|^{2(\gamma-1)}, |A^{1/2}u|^{2(\gamma-1)}\right\} \cdot \left||A^{1/2}u_\varepsilon|^2 - |A^{1/2}u|^2\right|. \tag{3.56}$$

From (3.11) and (3.18) we have that

$$\max\left\{|A^{1/2}u_\varepsilon(t)|^{2(\gamma-1)}, |A^{1/2}u(t)|^{2(\gamma-1)}\right\} \leq \frac{k_{14}}{(1+t)^{1-1/\gamma}}. \tag{3.57}$$

Moreover we have that

$$\begin{aligned} \left||A^{1/2}u_\varepsilon(t)|^2 - |A^{1/2}u(t)|^2\right| &= |\langle A(u_\varepsilon(t) + u(t)), u_\varepsilon(t) - u(t)\rangle| \\ &\leq \big(|Au_\varepsilon(t)| + |Au(t)|\big) \cdot |\rho_\varepsilon(t)|, \end{aligned}$$

so that from (3.12) and (3.19) we obtain that

$$\left||A^{1/2}u_\varepsilon(t)|^2 - |A^{1/2}u(t)|^2\right| \leq k_{15}\frac{|\rho_\varepsilon(t)|}{(1+t)^{1/(2\gamma)}}. \tag{3.58}$$

From (3.56), (3.57), (3.58), and (3.29), we conclude that

$$|c_\varepsilon(t) - c(t)| \leq k_{16}\frac{\varepsilon}{(1+t)^{1+(\delta-1)/(2\gamma)}}.$$

From (3.12) we have therefore that

$$\begin{aligned} |g_\varepsilon(t)|^2 &\leq 2(c_\varepsilon(t) - c(t))^2|Au(t)|^2 + 2\varepsilon^2|u''(t)|^2 \\ &\leq k_{17}\frac{\varepsilon^2}{(1+t)^{2+\delta/\gamma}} + 2\varepsilon^2|u''(t)|^2. \end{aligned} \tag{3.59}$$

At this point (3.30) follows from (3.15). Moreover (3.31) follows in an analogous way exploiting (3.14) instead of (3.12), and (3.16) instead of (3.15).

Finally, from (3.59) and (3.17) we obtain that

$$|g_\varepsilon(t)|^2 \leq k_{17}\frac{\varepsilon^2}{(1+t)^{2+\delta/\gamma}} + k_{18}\frac{\varepsilon^2}{(1+t)^{4+1/\gamma}} \leq k_{19}\frac{\varepsilon^2}{(1+t)^{2+\delta/\gamma}},$$

where in the last inequality we used that $2 + \delta/\gamma \leq 4 + 1/\gamma$. This proves (3.32), and completes the proof of Theorem 2.2. $\Box$

3.4 Proof of Theorem 2.3

The assumptions on initial data guarantee that the solution $u_\varepsilon(t)$ of the hyperbolic problem has only two Fourier components, and can be written in the form

$$u_\varepsilon(t) := u_{\varepsilon,\nu}(t)\frac{u_1}{|u_1|} + u_{\varepsilon,\mu}(t)\frac{u_0}{|u_0|},$$

where the coefficients $u_{\varepsilon,\nu}(t)$ and $u_{\varepsilon,\mu}(t)$ are given by (3.9) (note that in this case $u_0 = v_0$ and $u_1 = v_1$).

On the other hand, the solution of the parabolic problem has only the Fourier component with respect to u_0. Since u_0 and u_1 are orthogonal, we can estimate the norm of $\rho_\varepsilon(t)$ with the absolute value of its component with respect to u_1, namely

$$|\rho_\varepsilon(t)|^2 \geq \frac{\langle \rho_\varepsilon(t), u_1\rangle^2}{|u_1|^2} = \frac{\langle u_\varepsilon(t), u_1\rangle^2}{|u_1|^2} = |u_{\varepsilon,\nu}(t)|^2.$$

Therefore (2.3) is proved if we show that

$$\sup_{t\geq 0}\left\{(1+t)^{\delta/\gamma}|u_{\varepsilon,\nu}(t)|^2\right\} \geq k_1\varepsilon^2. \tag{3.60}$$

Estimate of $u_{\varepsilon,\nu}$ from above We prove that

$$|u_{\varepsilon,\nu}(t)| \leq k_2\varepsilon e^{-\nu C_\varepsilon(t)} \qquad \forall t\geq 0. \tag{3.61}$$

Let us consider in (3.10) the equation solved by $u_{\varepsilon,\nu}(t)$. Moving $\varepsilon u''_{\varepsilon,\nu}(t)$ to the right-hand side, we can interpret it as a first order equation. Since $u_{\varepsilon,\nu}(0) = 0$, integrating this differential equation we obtain that

$$u_{\varepsilon,\nu}(t) = -\varepsilon e^{-\nu C_\varepsilon(t)}\int_0^t u''_{\varepsilon,\nu}(s)e^{\nu C_\varepsilon(s)}\,ds.$$

Integrating by parts, and remarking that $u'_{\varepsilon,\nu}(0) = |u_1|$, this can be rewritten as

$$u_{\varepsilon,\nu}(t) = -\varepsilon u'_{\varepsilon,\nu}(t) + \varepsilon|u_1|e^{-\nu C_\varepsilon(t)} + \varepsilon e^{-\nu C_\varepsilon(t)}\int_0^t u'_{\varepsilon,\nu}(s)\nu c_\varepsilon(s)e^{\nu C_\varepsilon(s)}\,ds, \tag{3.62}$$

hence

$$|u_{\varepsilon,\nu}(t)| \leq \varepsilon e^{-\nu C_\varepsilon(t)}\left\{|u'_{\varepsilon,\nu}(t)|\cdot e^{\nu C_\varepsilon(t)} + |u_1| + \nu\int_0^t |u'_{\varepsilon,\nu}(s)|\cdot c_\varepsilon(s)e^{\nu C_\varepsilon(s)}\,ds\right\}.$$

Now from (3.27), and the estimate from above in (3.21), we have that

$$|u'_{\varepsilon,\nu}(t)|\cdot e^{\nu C_\varepsilon(t)} \leq k_3 c_\varepsilon(t) \leq \frac{k_4}{1+t}, \tag{3.63}$$

hence

$$|u_{\varepsilon,\nu}(t)| \leq \varepsilon e^{-\nu C_\varepsilon(t)}\left\{\frac{k_4}{1+t} + |u_1| + k_5\int_0^t \frac{1}{(1+s)^2}\,ds\right\},$$

which easily implies (3.61).

Estimate of $u_{\varepsilon,\nu}$ from below We prove that

$$u_{\varepsilon,\nu}(t) \geq \varepsilon e^{-\nu C_\varepsilon(t)} \left\{ |u_1| - \frac{k_6}{1+t} - k_7\varepsilon \right\} \qquad \forall t \geq 0. \tag{3.64}$$

To this end, we need to estimate the absolute value of the last term in (3.62). Thus we first integrate by parts, and we obtain that

$$\begin{aligned} \int_0^t u'_{\varepsilon,\nu}(s)\nu c_\varepsilon(s) e^{\nu C_\varepsilon(s)}\, ds &= u_{\varepsilon,\nu}(t)\nu c_\varepsilon(t) e^{\nu C_\varepsilon(t)} - \int_0^t u_{\varepsilon,\nu}(s)\nu c'_\varepsilon(s) e^{\nu C_\varepsilon(s)}\, ds \\ &\quad - \int_0^t u_{\varepsilon,\nu}(s)\nu^2 c_\varepsilon^2(s) e^{\nu C_\varepsilon(s)}\, ds \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

From (3.61), and the estimate from above in (3.21), we have that

$$|A_1| \leq k_8\varepsilon. \tag{3.65}$$

In order to control A_2 and A_3, we estimate $|u_{\varepsilon,\nu}(s)|$ by means of (3.61), we estimate $c_\varepsilon(s)$ by means of (3.21), and $|c'_\varepsilon(s)|$ by exploiting (3.21) and (3.22) in order to deduce that

$$|c'_\varepsilon(s)| = \frac{|c'_\varepsilon(s)|}{c_\varepsilon(s)} \cdot c_\varepsilon(s) \leq \frac{k_9}{(1+s)^2}.$$

Thus we obtain that

$$|A_2| + |A_3| \leq k_{10}\varepsilon \int_0^t \frac{1}{(1+s)^2}\, ds \leq k_{10}\varepsilon. \tag{3.66}$$

Plugging (3.65) and (3.66) into (3.62), and exploiting (3.63), we obtain that

$$\begin{aligned} u_{\varepsilon,\nu}(t) &\geq -\varepsilon|u'_{\varepsilon,\nu}(t)| + \varepsilon|u_1| e^{-\nu C_\varepsilon(t)} - k_{11}\varepsilon^2 e^{-\nu C_\varepsilon(t)} \\ &= \varepsilon e^{-\nu C_\varepsilon(t)} \left\{ |u_1| - |u'_{\varepsilon,\nu}(t)| e^{\nu C_\varepsilon(t)} - k_{11}\varepsilon \right\} \\ &\geq \varepsilon e^{-\nu C_\varepsilon(t)} \left\{ |u_1| - \frac{k_4}{1+t} - k_{11}\varepsilon \right\}, \end{aligned}$$

which is exactly (3.64).

Estimate of the exponential Let us set for simplicity $\psi_\varepsilon(t) := e^{-\nu C_\varepsilon(t)}$. We prove that

$$\psi_\varepsilon\left(\frac{1}{\varepsilon^\delta}\right) \geq k_{12}\varepsilon^{\delta^2/(2\gamma)}. \tag{3.67}$$

Indeed from (3.61) and (3.28) we have that

$$|A^{1/2}u_\varepsilon(t)|^2 = \nu|u_{\varepsilon,\nu}(t)|^2 + \mu|u_{\varepsilon,\mu}(t)|^2 \leq k_{13}\left(\varepsilon^2[\psi_\varepsilon(t)]^2 + [\psi_\varepsilon(t)]^{2\mu/\nu}\right),$$

hence

$$c_\varepsilon(t) = |A^{1/2}u_\varepsilon(t)|^{2\gamma} \leq k_{14}\left(\varepsilon^{2\gamma}[\psi_\varepsilon(t)]^{2\gamma} + [\psi_\varepsilon(t)]^{2\gamma/\delta}\right).$$

Since $\psi_\varepsilon'(t) = -\nu c_\varepsilon(t)\psi_\varepsilon(t)$, this implies that

$$\psi_\varepsilon'(t) \geq -k_{15}\psi_\varepsilon(t)\left(\varepsilon^{2\gamma}[\psi_\varepsilon(t)]^{2\gamma} + [\psi_\varepsilon(t)]^{2\gamma/\delta}\right) \qquad \forall t \geq 0.$$

At this point (3.67) follows from Lemma 3.2.

Conclusion We are now ready to prove (3.60). Let us set $t := 1/\varepsilon^\delta$ in (3.64). We obtain that

$$u_{\varepsilon,\nu}\left(\frac{1}{\varepsilon^\delta}\right) \geq \varepsilon\psi_\varepsilon\left(\frac{1}{\varepsilon^\delta}\right)\left\{|u_1| - k_6\frac{\varepsilon^\delta}{1+\varepsilon^\delta} - k_7\varepsilon\right\}.$$

If ε is small enough, the term between braces is larger than or equal to $|u_1|/2$, hence by (3.67) we obtain that

$$u_{\varepsilon,\nu}\left(\frac{1}{\varepsilon^\delta}\right) \geq \frac{|u_1|}{2}\varepsilon\psi_\varepsilon\left(\frac{1}{\varepsilon^\delta}\right) \geq k_{16}\varepsilon \cdot \varepsilon^{\delta^2/(2\gamma)}.$$

Therefore we conclude that

$$\sup_{t\geq 0}|u_{\varepsilon,\nu}(t)|^2(1+t)^{\delta/\gamma} \geq \left|u_{\varepsilon,\nu}\left(\frac{1}{\varepsilon^\delta}\right)\right|^2\left(\frac{1+\varepsilon^\delta}{\varepsilon^\delta}\right)^{\delta/\gamma} \geq k_{17}\varepsilon^2,$$

which completes the proof. □

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